

## Chapter 2. Electrostatic II

### Notes:

- Most of the material presented in this chapter is taken from Jackson, Chap. 2, 3, and 4, and Di Bartolo, Chap. 2.

### 2.1 Mathematical Considerations

#### 2.1.1 The Fourier series and the Fourier transform

Any periodic signal  $g(x)$ , of period  $a$ , can always be expressed with the so-called **Fourier series**, where

$$g(x) = \sum_{n=-\infty}^{\infty} G_n e^{i \frac{2\pi n x}{a}}, \quad (2.1)$$

with  $G_n$  the Fourier coefficient

$$G_n = \frac{1}{a} \int_{-a/2}^{a/2} g(x) e^{-i \frac{2\pi n x}{a}} dx. \quad (2.2)$$

Alternatively, the Fourier series can be expressed using sine and cosine functions instead of an exponential function. Equations (2.1) and (2.2) are then replaced by

$$\begin{aligned} g(x) &= G_0 + \sum_{n=1}^{\infty} \left[ A_n \cos\left(\frac{2\pi n x}{a}\right) + B_n \sin\left(\frac{2\pi n x}{a}\right) \right] \\ A_n &= \frac{2}{a} \int_{-a/2}^{a/2} g(x) \cos\left(\frac{2\pi n x}{a}\right) dx \\ B_n &= \frac{2}{a} \int_{-a/2}^{a/2} g(x) \sin\left(\frac{2\pi n x}{a}\right) dx. \end{aligned} \quad (2.3)$$

Note that  $G_0 = A_0/2$ , if  $n = 0$  is allowed in the second of equations (2.3). If we set

$$g(x) = \sum_{m=-\infty}^{\infty} \delta(x - x' - ma), \quad \text{for } |x'| < \frac{a}{2} \quad (2.4)$$

then it seen from equations (2.1) and (2.2) that

$$\delta(x - x') = \frac{1}{a} \sum_{n=-\infty}^{\infty} e^{i \frac{2\pi n (x-x')}{a}}, \quad \text{for } |x| < \frac{a}{2}. \quad (2.5)$$

This is the *closure relation*.

If we let  $a \rightarrow \infty$ , we have the following transformations

$$\begin{aligned} \frac{2\pi n}{a} &\rightarrow k \\ \sum_{n=-\infty}^{\infty} &\rightarrow \int_{-\infty}^{\infty} dn = \frac{a}{2\pi} \int_{-\infty}^{\infty} dk \\ G_n &\rightarrow \frac{2\pi}{a} G(k). \end{aligned} \quad (2.6)$$

The pair of equations (2.1) and (2.2) defining the Fourier series are replaced by a corresponding set

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} G(k) e^{ikx} dk \\ G(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-ikx} dx. \end{aligned} \quad (2.7)$$

These equations are usually rendered symmetric by, respectively, dividing and multiplying them by  $\sqrt{2\pi}$ . We then get the **Fourier transform pair**

$$\begin{aligned} g(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(k) e^{ikx} dk \\ G(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-ikx} dx. \end{aligned} \quad (2.8)$$

We can evaluate the closure relation corresponding to the Fourier transform by setting  $g(x) = \delta(x - x')$  in equations (2.8), we thus obtain

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk. \quad (2.9)$$

The generalization of both the Fourier series and Fourier transform to functions of higher dimensions is straightforward.

### 2.1.2 Spherical harmonics

One possible solution to the Laplace equation is a potential function of polynomials of rank  $l$ . If we use Cartesian coordinates we can write this potential as

$$\Phi_l(\mathbf{x}) = \sum_{n_1+n_2+n_3=l} C_{n_1 n_2 n_3} x^{n_1} y^{n_2} z^{n_3}. \quad (2.10)$$

We now evaluate the maximum number of terms in the summation. If we start by assigning a value  $m$  (one out of the  $l+1$  possible values) to the exponent  $n_1$ , we can easily determine that there are  $l+1-m$  different combinations with which we can arrange the other two indices. If we denote by  $N(l)$  the maximum number of terms composing the potential, we have

$$N(l) = \sum_{m=0}^l (l+1-m) = \sum_{m=1}^{l+1} (l+2-m). \quad (2.11)$$

But since this equation can also simply be written as

$$N(l) = \sum_{m=1}^{l+1} m, \quad (2.12)$$

we can transform equation (2.11) as follows

$$\begin{aligned} N(l) &= (l+2)(l+1) - \sum_{m=1}^{l+1} m \\ &= (l^2 + 3l + 2) - N(l). \end{aligned} \quad (2.13)$$

Then,

$$N(l) = \frac{1}{2}(l^2 + 3l + 2). \quad (2.14)$$

However, these  $N(l)$  terms cannot all be independent since the potential must satisfy the Laplace equation, which states that

$$\nabla^2 \Phi_l = 0. \quad (2.15)$$

Because this last equation has a rank of  $l-2$ , the number of independent terms is given by

$$\begin{aligned} N(l) - N(l-2) &= \frac{1}{2}[l^2 + 3l + 2] - \frac{1}{2}[(l-2)^2 + 3(l-2) + 2] \\ &= 2l + 1. \end{aligned} \quad (2.16)$$

If we make a change of coordinates from Cartesian to spherical with

$$\begin{aligned}
x &= r \sin(\theta) \cos(\varphi) \\
y &= r \sin(\theta) \sin(\varphi) \\
z &= r \cos(\theta),
\end{aligned} \tag{2.17}$$

and further

$$\begin{aligned}
x + iy &= r \sin(\theta) e^{i\varphi} \\
x - iy &= r \sin(\theta) e^{-i\varphi},
\end{aligned} \tag{2.18}$$

we can write the potential as

$$\Phi_l(r, \theta, \varphi) = \sum_{m=-l}^l C_m r^l Y_{lm}(\theta, \varphi), \tag{2.19}$$

where the functions  $Y_{lm}(\theta, \varphi)$ , with  $-l < m < l$ , are the **spherical harmonics**.

The Laplacian operator in spherical coordinates is given by

$$\begin{aligned}
\nabla^2 &= \frac{1}{r} \frac{\partial^2}{\partial r^2} r + \frac{1}{r^2} \left\{ \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left[ \sin(\theta) \frac{\partial}{\partial \theta} \right] + \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial \varphi^2} \right\} \\
&= \frac{1}{r} \frac{\partial^2}{\partial r^2} r + \frac{\Omega}{r^2},
\end{aligned} \tag{2.20}$$

with

$$\Omega = \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left[ \sin(\theta) \frac{\partial}{\partial \theta} \right] + \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial \varphi^2}. \tag{2.21}$$

Applying the Laplace equation to the term  $(l, m)$  of the potential using the last of equations (2.20) yields

$$\begin{aligned}
\left[ \frac{1}{r} \frac{\partial^2}{\partial r^2} r + \frac{\Omega}{r^2} \right] r^l Y_{lm} &= \frac{1}{r} \frac{\partial^2}{\partial r^2} (r^{l+1} Y_{lm}) + \Omega r^{l-2} Y_{lm} \\
&= l(l+1) r^{l-2} Y_{lm} + r^{l-2} \Omega Y_{lm} = 0.
\end{aligned} \tag{2.22}$$

Thus, the differential equation satisfied by the spherical harmonics is

$$\Omega Y_{lm}(\theta, \varphi) + l(l+1) Y_{lm}(\theta, \varphi) = 0. \tag{2.23}$$

Equation (2.23) is commonly known as the **generalized Legendre differential equation**. Because the coordinates can be separated from each other in the Laplace equation, we

can better establish a more general functional form for the solution to the Laplace equation in spherical coordinates. For example, if we act on

$$\Phi(r, \theta, \varphi) = \frac{U(r)}{r} P(\theta) Q(\varphi), \quad (2.24)$$

with equation (2.20) (while setting the result of the operation equal to zero), multiply by  $r^2 \sin^2(\theta)$ , and divide by equation (2.24), we find that  $\varphi$  is separated from  $r$  and  $\theta$  and consequently

$$\frac{1}{Q} \frac{d^2 Q}{d\varphi^2} = -m^2. \quad (2.25)$$

with  $m$  a constant. The solution to equation (2.25) is

$$Q(\varphi) \propto e^{im\varphi}. \quad (2.26)$$

If we insert equations (2.24) and (2.26) into equation (2.20), multiply by  $r^2$ , and divide by equation (2.24),  $r$  is now separated from  $\theta$  and  $\varphi$  with

$$\frac{r^2}{U} \frac{d^2 U}{dr^2} = l(l+1), \quad (2.27)$$

with  $l$  a constant, and therefore

$$\frac{U(r)}{r} = A_l r^l + \frac{B_l}{r^{l+1}} \quad (2.28)$$

in general. Finally, if we insert equations (2.24), (2.26), and (2.28) in equation (2.20), multiply by  $r^2$ , and divide by equation (2.24), then  $\theta$  is separated from  $r$  and  $\varphi$  with

$$\frac{1}{\sin(\theta)} \frac{d}{d\theta} \left[ \sin(\theta) \frac{dP}{d\theta} \right] + \left[ l(l+1) - \frac{m^2}{\sin^2(\theta)} \right] P = 0, \quad (2.29)$$

and a corresponding solution  $P_m(\theta)$ . The general solution to the Laplace equation in spherical coordinates is therefore

$$\Phi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left[ A_{lm} r^l + \frac{B_{lm}}{r^{l+1}} \right] Y_{lm}(\theta, \varphi), \quad (2.30)$$

with

$$Y_{lm}(\theta, \varphi) \propto P_l^m(\theta) e^{-im\varphi}. \quad (2.31)$$

The spherical harmonics also satisfy the following relations:

**a)** The *orthonormality relation*

$$\int_0^{2\pi} d\varphi \int_0^\pi Y_{lm}^*(\theta, \varphi) Y_{l'm'}(\theta, \varphi) \sin(\theta) d\theta = \delta_{ll'} \delta_{mm'}. \quad (2.32)$$

**b)** If we write

$$Y_{lm}(\theta, \varphi) = Y_{lm}\left(\frac{\mathbf{r}}{r}\right) = Y_{lm}(\mathbf{n}), \quad (2.33)$$

where  $-\mathbf{n}$  has the orientation  $\pi - \theta$  and  $\pi + \varphi$ , then the following *symmetry relation* applies

$$Y_{lm}(-\mathbf{n}) = (-1)^l Y_{lm}(\mathbf{n}). \quad (2.34)$$

**c)** If we substitute  $m \rightarrow -m$

$$Y_{l,-m}(\theta, \varphi) = (-1)^m Y_{lm}^*(\theta, \varphi). \quad (2.35)$$

**d)** The *closure relation*

$$\sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) = \delta(\varphi - \varphi') \delta(\cos(\theta) - \cos(\theta')). \quad (2.36)$$

Since the spherical harmonics form a complete set of orthonormal functions, any arbitrary function can be expanded as a series of spherical harmonics with

$$g(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l A_{lm} Y_{lm}(\theta, \varphi) \quad (2.37)$$

$$A_{lm} = \int_0^{2\pi} d\varphi \int_0^\pi g(\theta, \varphi) Y_{lm}^*(\theta, \varphi) \sin(\theta) d\theta.$$

Here are some examples of spherical harmonics

$$\begin{aligned}
Y_{00} &= \frac{1}{\sqrt{4\pi}} \\
Y_{10} &= \sqrt{\frac{3}{4\pi}} \cos(\theta) \\
Y_{1,\pm 1} &= \mp \sqrt{\frac{3}{8\pi}} \sin(\theta) e^{\pm i\varphi} \\
Y_{20} &= \sqrt{\frac{5}{16\pi}} [3\cos^2(\theta) - 1] \\
Y_{2,\pm 1} &= \mp \sqrt{\frac{15}{8\pi}} \sin(\theta) \cos(\theta) e^{\pm i\varphi} \\
Y_{2,\pm 2} &= \sqrt{\frac{15}{32\pi}} \sin^2(\theta) e^{\pm i2\varphi}.
\end{aligned} \tag{2.38}$$

The **Legendre polynomials** are the solution to the generalized Legendre differential equation (i.e., equation (2.23)) when  $m=0$ . They can be defined using the spherical harmonics as

$$P_l[\cos(\theta)] = \sqrt{\frac{4\pi}{2l+1}} Y_{l0}(\theta). \tag{2.39}$$

The Legendre polynomials also form a complete set of orthogonal functions with an *orthogonality relation*

$$\int_{-1}^1 P_{l'}(x) P_l(x) dx = \frac{2}{2l+1} \delta_{l'l} \tag{2.40}$$

and a *closure relation*

$$\sum_{l=0}^{\infty} P_l^*(x') P_l(x) = \frac{2}{2l+1} \delta(x-x'), \tag{2.41}$$

where  $x = \cos(\theta)$ . Any arbitrary function of  $x = \cos(\theta)$  can, therefore, be expanded in a series of Legendre Polynomials

$$\begin{aligned}
g(x) &= \sum_{l=0}^{\infty} A_l P_l(x) \\
A_l &= \frac{2l+1}{2} \int_{-1}^1 g(x) P_l(x) dx.
\end{aligned} \tag{2.42}$$

The first few Legendre polynomials are

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2 - 1) \\
P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3).
\end{aligned} \tag{2.43}$$

There exists a useful relation between the Legendre polynomials and the spherical harmonics. The so-called **addition theorem for spherical harmonics** is expressed as follows

$$\boxed{P_l[\cos(\gamma)] = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi)} \tag{2.44}$$

where  $\gamma$  is the angle made between two vectors  $\mathbf{x}$  and  $\mathbf{x}'$  of coordinates  $(r, \theta, \varphi)$  and  $(r', \theta', \varphi')$ , respectively. These definitions for  $\mathbf{x}$  and  $\mathbf{x}'$  also imply that

$$\cos(\gamma) = \cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\varphi - \varphi'). \tag{2.45}$$

We can prove the addition theorem by noting that since  $\cos(\gamma)$  is a function of  $\theta$  and  $\varphi$  through equation (2.45), then,  $P_l[\cos(\gamma)]$  can be expanded in a series of spherical harmonics  $Y_{lm}(\theta, \varphi)$ . We therefore write

$$P_l[\cos(\gamma)] = \sum_{m=-l}^l c_m Y_{lm}(\theta, \varphi). \tag{2.46}$$

Evidently, since equation (2.45) is symmetric in  $(\theta, \varphi)$  and  $(\theta', \varphi')$ , we could just as well express  $P_l[\cos(\gamma)]$  with a similar series containing  $\theta'$  and  $\varphi'$  instead

$$P_l[\cos(\gamma)] = \sum_{m'=-l}^l c'_{m'} Y_{lm'}(\theta', \varphi'). \tag{2.47}$$

Furthermore, because of the particular form of equation (2.45) we can also write

$$P_l[\cos(\gamma)] = \sum_{m=-l}^l \sum_{m'=-l}^l \alpha_{mm'} Y_{lm'}(\theta', \varphi') Y_{lm}(\theta, \varphi). \tag{2.48}$$

[**Note:** Equation (2.48) is justified because each term in  $\cos(\gamma)$  can be expressed by a sum of products of functions of  $\theta$  and  $\varphi$  with functions of  $\theta'$  and  $\varphi'$  through

$$\begin{aligned} \cos(\gamma) = & \sin(\theta)\cos(\varphi)\sin(\theta')\cos(\varphi') + \sin(\theta)\sin(\varphi)\sin(\theta')\sin(\varphi') \\ & + \cos(\theta)\cos(\theta'), \end{aligned} \quad (2.49)$$

and therefore the same can be said of  $P_l[\cos(\gamma)]$ , since it contains a sum of terms of the type  $\cos^n(\gamma)$  (which can similarly be expressed as a sum of products of functions of  $\theta$  and  $\varphi$  with functions of  $\theta'$  and  $\varphi'$ ). For a given  $l$ , an arbitrary function  $f_l(\theta, \varphi)$  ( $f_l(\theta', \varphi')$ ) can be expanded as a series of spherical harmonics  $Y_{lm}(\theta, \varphi)$  ( $Y_{lm'}(\theta', \varphi')$ ); ergo equation (2.48).]

However, because of equations (2.31) and (2.45) the dependence on  $\varphi$  must be such that

$$e^{im'\varphi'} e^{im\varphi} = e^{i(m'\varphi' + m\varphi)} = e^{im(\varphi - \varphi')}. \quad (2.50)$$

That is, we must have  $m' = -m$ , and

$$P_l[\cos(\gamma)] = \sum_{m=-l}^l \alpha_m Y_{l,-m}(\theta', \varphi') Y_{lm}(\theta, \varphi). \quad (2.51)$$

Substituting equation (2.35) for the first spherical harmonic, we get

$$P_l[\cos(\gamma)] = \sum_{m=-l}^l (-1)^m \alpha_m Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi). \quad (2.52)$$

If we now set  $\gamma = 0$  so that  $\cos(\gamma) = 1$ ,  $\theta' = \theta$ , and  $\varphi' = \varphi$ , we have from equation (2.52)

$$P_l(1) = 1 = \sum_{m=-l}^l (-1)^m \alpha_m |Y_{lm}(\theta, \varphi)|^2, \quad (2.53)$$

and integrating both sides over all angles we find, because the normalization of the spherical harmonics (see equation (2.32)),

$$4\pi = \sum_{m=-l}^l (-1)^m \alpha_m. \quad (2.54)$$

Let us now square equation (2.52)

$$\begin{aligned}
P_l^2[\cos(\gamma)] &= \sum_{m=-l}^l (-1)^m \alpha_m Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) \\
&\quad \times \sum_{m'=-l}^l (-1)^{m'} \alpha_{m'} Y_{lm'}^*(\theta', \varphi') Y_{lm'}(\theta, \varphi) \\
&= \sum_{m=-l}^l (-1)^m \alpha_m Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) \\
&\quad \times \sum_{m'=-l}^l (-1)^{m'} \alpha_{m'} Y_{lm'}(\theta', \varphi') Y_{lm'}^*(\theta, \varphi),
\end{aligned} \tag{2.55}$$

where we have used equation (2.35) twice in the last term, and made the transformations  $m' \rightarrow -m'$  and  $\sum_{m'} \rightarrow \sum_{-m'}$  as well. We integrate over the angles  $\theta$  and  $\varphi$  to obtain

$$\begin{aligned}
\int P_l^2[\cos(\gamma)] d\Omega &= \sum_{m=-l}^l \sum_{m'=-l}^l (-1)^{m+m'} \alpha_m \alpha_{m'} Y_{lm}^*(\theta', \varphi') Y_{lm'}(\theta', \varphi') \\
&\quad \times \int Y_{lm'}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) d\Omega,
\end{aligned} \tag{2.56}$$

which according to equations (2.32) and (2.40) gives

$$\frac{4\pi}{2l+1} = \sum_{m=-l}^l \alpha_m^2 |Y_{lm}(\theta', \varphi')|^2. \tag{2.57}$$

Integrating over the angles  $\theta'$  and  $\varphi'$  this time, we get

$$4\pi \frac{4\pi}{2l+1} = \sum_{m=-l}^l \alpha_m^2. \tag{2.58}$$

Combining equations (2.54) and (2.58) we can determine the coefficient  $\alpha_m$  to be

$$\alpha_m = \frac{4\pi(-1)^m}{2l+1}, \tag{2.59}$$

which upon insertion in equation (2.52) yield equation (2.44), i.e., the addition theorem for spherical harmonics.

Finally, another useful equation is that for the expansion of the potential of point charge, in a volume excluding the charge (i.e.,  $\mathbf{x} \neq \mathbf{x}'$ ), as a function of Legendre polynomials

$$\boxed{\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{l=0}^{\infty} \frac{r_{<}^l}{r_{>}^{l+1}} P_l[\cos(\gamma)]}, \tag{2.60}$$

where  $r_<$  and  $r_>$  are, respectively, the smaller and larger of  $|\mathbf{x}|$  and  $|\mathbf{x}'|$ , and  $\gamma$  is the angle between  $\mathbf{x}$  and  $\mathbf{x}'$  (see equation (2.45)). Upon substituting the addition theorem of equation (2.44) for the Legendre polynomials in equation (2.60), we find

$$\boxed{\frac{1}{|\mathbf{x} - \mathbf{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi)}. \quad (2.61)$$

## 2.2 Solution of Electrostatic Boundary-value Problems

As we saw earlier, the expression for the potential obtained through Green's theorem is a solution of the Poisson equation

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}. \quad (2.62)$$

It is always possible to write  $\Phi$  as the superposition of two potentials: the particular solution  $\Phi_p$ , and the characteristic solution  $\Phi_c$

$$\Phi = \Phi_p + \Phi_c. \quad (2.63)$$

The characteristic potential is actually the solution to the homogeneous Laplace equation

$$\nabla^2 \Phi = 0 \quad (2.64)$$

due to the boundary conditions on the surface  $S$  that delimitates the volume  $V$  where the potential is evaluated (i.e.,  $\Phi_c$  is the result of the surface integral in equation (1.106) after the desired type of boundary conditions are determined). Consequently,  $\Phi_p$  is the response to the presence of the charges present within  $V$ . More precisely,

$$\Phi_p(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{x}')}{R} d^3x'. \quad (2.65)$$

This suggests that we can always break down a boundary-value problem into two parts. To the particular solution that is formally evaluated with equation (2.65), we add the characteristic potential, which is often more easily solved using the Laplace equation (instead of the aforementioned surface integral).

### 2.2.1 Separation of variables in Cartesian coordinates

The Laplace equation in Cartesian coordinates is given by

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0, \quad (2.66)$$

where we dropped the subscript “c” for the potential as it is understood that we are solving for the characteristic potential. A general form for the solution to this equation can be attempted by assuming that the potential can be expressed as the product of three functions, each one being dependent of one variable only. More precisely,

$$\Phi(x, y, z) = X(x)Y(y)Z(z). \quad (2.67)$$

Inserting this relation into equation (2.66), and dividing the result by equation (2.67) we get

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} + \frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = 0. \quad (2.68)$$

If this equation is to hold for any values of  $x, y$ , and  $z$ , then each term must be constant. We, therefore, write

$$\begin{aligned} \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} &= -\alpha^2 \\ \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} &= -\beta^2 \\ \frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} &= \gamma^2, \end{aligned} \quad (2.69)$$

with

$$\alpha^2 + \beta^2 = \gamma^2. \quad (2.70)$$

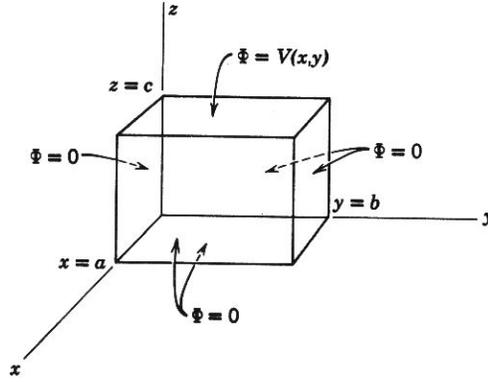
The solution to equations (2.69), when  $\alpha^2$  and  $\beta^2$  are both positive, is of the type

$$\Phi(x, y, z) \propto e^{\pm i\alpha x} e^{\pm i\beta y} e^{\pm \sqrt{\alpha^2 + \beta^2} z}. \quad (2.71)$$

### Example

Let’s consider a hollow rectangular box of dimension  $a, b$  and  $c$  in the  $x, y$ , and  $z$  directions, respectively, with five of the six sides kept at zero potential (grounded) and the top side at a voltage  $V(x, y)$ . The box has one of its corners located at the origin (see Figure 2.1). We want to evaluate the potential everywhere inside the box.

Since we have  $\Phi = 0$  for  $x = 0$ ,  $y = 0$ , and  $z = 0$ , it must be that



**Figure 2.1** – A hollow, rectangular box with five sides at zero potential. The topside has the specified voltage  $\Phi = V(x, y)$ .

$$\begin{aligned}
 X &\propto \sin(\alpha x) \\
 Y &\propto \sin(\beta y) \\
 Z &\propto \sinh\left(\sqrt{\alpha^2 + \beta^2} z\right).
 \end{aligned}
 \tag{2.72}$$

Moreover, since  $\Phi = 0$  at  $x = a$ , and  $y = b$ , we must further have that

$$\begin{aligned}
 \alpha_n &= \frac{n\pi}{a} \\
 \beta_m &= \frac{m\pi}{b} \\
 \gamma_{nm} &= \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}},
 \end{aligned}
 \tag{2.73}$$

where the subscripts  $n$  and  $m$  were added to identify the different “modes” allowed to exist in the box. To each mode  $nm$  corresponds a potential

$$\Phi_{nm} \propto \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z).
 \tag{2.74}$$

The total potential will consist of a linear superposition of the potentials belonging to the different modes

$$\Phi(x, y, z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z),
 \tag{2.75}$$

where the still undetermined coefficient  $A_{nm}$  is the amplitude of the potential associated with mode  $nm$ . Finally, we must also take into account the value of the potential at  $z = c$  with

$$V(x, y) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c). \quad (2.76)$$

We see from equation (2.76) that  $A_{nm} \sinh(\gamma_{nm} c)$  is just a (two-dimensional) Fourier coefficient, and is given by

$$A_{nm} \sinh(\gamma_{nm} c) = \frac{4}{ab} \int_0^a dx \int_0^b dy V(x, y) \sin(\alpha_n x) \sin(\beta_m y). \quad (2.77)$$

If, for example, the potential is kept constant at  $V$  over the topside, then the coefficient is given by

$$A_{nm} \sinh(\gamma_{nm} c) = V \frac{4}{\pi^2 nm} [1 - (-1)^n] [1 - (-1)^m]. \quad (2.78)$$

We see that no even modes are allowed, and that the amplitude of a given mode is inversely proportional to its order.

### 2.3 Multipole Expansion

Given a charge distribution  $\rho(\mathbf{x}')$  contained within a sphere of radius  $R$ , we want to evaluate the potential  $\Phi(\mathbf{x})$  at any point exterior to the sphere. Since the potential is evaluated at points where there are no charges, it must satisfy the Laplace equation and, therefore, can be expanded as a series of spherical harmonics

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^l c_{lm} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}}, \quad (2.79)$$

where the coefficients  $c_{lm}$  are to be determined. The radial functional form chosen for the potential in equation (2.79) is the only physical possibility available from the general solution to the Laplace equation (see equation (2.30)), as the potential must be finite when  $r \rightarrow \infty$ . In order to evaluate the coefficients  $c_{lm}$ , we use the well-known volume integral for the potential

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x', \quad (2.80)$$

with equation (2.61) for the expansion of the denominator. We, then, find (with  $r_{<} = r' = |\mathbf{x}'|$  and  $r_{>} = r = |\mathbf{x}|$ , since  $|\mathbf{x}| > |\mathbf{x}'|$ )

$$\Phi(\mathbf{x}) = \frac{1}{\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{2l+1} \left[ \int Y_{lm}^*(\theta', \varphi') r'^l \rho(\mathbf{x}') d^3x' \right] \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}}. \quad (2.81)$$

Comparing equations (2.79) and (2.81), we determine the coefficients of the former equation to be

$$c_{lm} = \frac{4\pi}{2l+1} \int Y_{lm}^*(\theta', \varphi') r'^l \rho(\mathbf{x}') d^3 x'. \quad (2.82)$$

The **multipole moments**, denoted by  $q_{lm} (= c_{lm} (2l+1)/4\pi)$ , are given by

$$q_{lm} = \int Y_{lm}^*(\theta', \varphi') r'^l \rho(\mathbf{x}') d^3 x' \quad (2.83)$$

Using some of equations (2.38), with

$$\begin{aligned} r' \sin(\theta') e^{\pm i\varphi} &= x' \pm iy' \\ r' \cos(\theta') &= z', \end{aligned} \quad (2.84)$$

we can calculate some of the moments, with for example

$$\begin{aligned} q_{00} &= \frac{1}{\sqrt{4\pi}} \int \rho(\mathbf{x}') d^3 x' = \frac{q}{\sqrt{4\pi}} \\ q_{10} &= \sqrt{\frac{3}{4\pi}} \int \cos(\theta) r' \rho(\mathbf{x}') d^3 x' = \sqrt{\frac{3}{4\pi}} \int z' \rho(\mathbf{x}') d^3 x' \\ &= \sqrt{\frac{3}{4\pi}} p_z \\ q_{1,\pm 1} &= \mp \sqrt{\frac{3}{8\pi}} \int \sin(\theta) e^{\mp i\varphi} r' \rho(\mathbf{x}') d^3 x' = \mp \sqrt{\frac{3}{8\pi}} \int (x' \mp iy') \rho(\mathbf{x}') d^3 x' \\ &= \mp \sqrt{\frac{3}{8\pi}} (p_x \mp ip_y) \end{aligned} \quad (2.85)$$

for  $l \leq 1$ , and the following for  $l = 2$

$$\begin{aligned}
q_{20} &= \sqrt{\frac{5}{16\pi}} \int [3\cos^2(\theta) - 1] r'^2 \rho(\mathbf{x}') d^3x' = \sqrt{\frac{5}{16\pi}} \int [3z'^2 - r'^2] \rho(\mathbf{x}') d^3x' \\
&= \sqrt{\frac{5}{16\pi}} Q_{33} \\
q_{2,\pm 1} &= \mp \sqrt{\frac{15}{8\pi}} \int \sin(\theta) \cos(\theta) e^{\mp i\varphi} r'^2 \rho(\mathbf{x}') d^3x' = \mp \sqrt{\frac{15}{8\pi}} \int z'(x' \mp iy') \rho(\mathbf{x}') d^3x' \\
&= \mp \frac{1}{3} \sqrt{\frac{15}{8\pi}} (Q_{13} \mp iQ_{23}) \\
q_{2,\pm 2} &= \sqrt{\frac{15}{32\pi}} \int \sin^2(\theta) e^{\mp i2\varphi} r'^2 \rho(\mathbf{x}') d^3x' = \sqrt{\frac{15}{32\pi}} \int (x' \mp iy')^2 \rho(\mathbf{x}') d^3x' \\
&= \frac{1}{3} \sqrt{\frac{15}{32\pi}} (Q_{11} \mp 2iQ_{12} - Q_{22}).
\end{aligned} \tag{2.86}$$

In equations (2.85) and (2.86),  $q$  is the total charge or **monopole moment**,  $\mathbf{p}$  is the **electric dipole moment**

$$\mathbf{p} = \int \mathbf{x}' \rho(\mathbf{x}') d^3x', \tag{2.87}$$

and  $Q_{ij}$  is a component of the **quadrupole moment tensor** (here, no summation is implied when  $i = j$ )

$$Q_{ij} = \int (3x'_i x'_j - r'^2 \delta_{ij}) \rho(\mathbf{x}') d^3x'. \tag{2.88}$$

Using a Taylor expansion (see equation (1.84)) for  $1/|\mathbf{x} - \mathbf{x}'|$ , we can also express the potential in rectangular coordinate (the proof will be found with the solution to the first problem list) with

$$\Phi(\mathbf{x}) \simeq \frac{1}{4\pi\epsilon_0} \left[ \frac{q}{r} + \frac{\mathbf{p} \cdot \mathbf{x}}{r^3} + \frac{1}{2} Q_{ij} \frac{x_i x_j}{r^5} + \dots \right]. \tag{2.89}$$

The electric field corresponding to a given multipole is given by

$$\mathbf{E} = -\nabla\Phi_{lm}, \tag{2.90}$$

and from equations (2.79), (2.82), and (2.83) we find

$$\begin{aligned}
E_r &= -\frac{\partial\Phi}{\partial r} = \frac{1}{\epsilon_0} \frac{(l+1)}{(2l+1)} q_{lm} \frac{Y_{lm}(\theta, \varphi)}{r^{l+2}} \\
E_\theta &= -\frac{1}{r} \frac{\partial\Phi}{\partial\theta} = -\frac{1}{\epsilon_0} \frac{1}{(2l+1)} q_{lm} \frac{1}{r^{l+2}} \frac{\partial}{\partial\theta} Y_{lm}(\theta, \varphi) \\
E_\varphi &= -\frac{1}{r \sin(\theta)} \frac{\partial\Phi}{\partial\varphi} = -\frac{1}{\epsilon_0} \frac{1}{(2l+1)} q_{lm} \frac{1}{r^{l+2}} \frac{im}{\sin(\theta)} Y_{lm}(\theta, \varphi).
\end{aligned} \tag{2.91}$$

For example, for the electric monopole term ( $l = 0$ ) we find

$$E_r = \frac{q}{4\pi\epsilon_0 r^2} \quad \text{and} \quad E_\theta = E_\varphi = 0, \tag{2.92}$$

which is equivalent to the field generated by a point charge  $q$ , as expected. For the electric dipole term ( $l = 1$ ), we have

$$\begin{aligned}
E_r &= \frac{2}{3\epsilon_0 r^3} [q_{1,-1} Y_{1,-1} + q_{10} Y_{10} + q_{11} Y_{11}] \\
&= \frac{2}{3\epsilon_0 r^3} \frac{3}{8\pi} [(p_x + ip_y) \sin(\theta) e^{-i\varphi} + 2p_z \cos(\theta) + (p_x - ip_y) \sin(\theta) e^{i\varphi}] \\
&= \frac{2}{4\pi\epsilon_0 r^3} [p_x \sin(\theta) \cos(\varphi) + p_y \sin(\theta) \sin(\varphi) + p_z \cos(\theta)] \\
E_\theta &= -\frac{1}{3\epsilon_0 r^3} \left[ q_{1,-1} \frac{\partial Y_{1,-1}}{\partial\theta} + q_{10} \frac{\partial Y_{10}}{\partial\theta} + q_{11} \frac{\partial Y_{11}}{\partial\theta} \right] \\
&= -\frac{1}{3\epsilon_0 r^3} \frac{3}{8\pi} [(p_x + ip_y) \cos(\theta) e^{-i\varphi} - 2p_z \sin(\theta) + (p_x - ip_y) \cos(\theta) e^{i\varphi}] \\
&= -\frac{1}{4\pi\epsilon_0 r^3} [p_x \cos(\theta) \cos(\varphi) + p_y \cos(\theta) \sin(\varphi) - p_z \sin(\theta)],
\end{aligned} \tag{2.93}$$

and

$$\begin{aligned}
E_\varphi &= -\frac{i}{3\epsilon_0 r^3 \sin(\theta)} [-q_{1,-1} Y_{1,-1} + q_{11} Y_{11}] \\
&= -\frac{i}{3\epsilon_0 r^3 \sin(\theta)} \frac{3}{8\pi} [-(p_x + ip_y) \sin(\theta) e^{-i\varphi} + (p_x - ip_y) \sin(\theta) e^{i\varphi}] \\
&= -\frac{1}{4\pi\epsilon_0 r^3} [-p_x \sin(\varphi) + p_y \cos(\varphi)].
\end{aligned} \tag{2.94}$$

Because the unit basis vectors of the spherical and Cartesian coordinates are related by the following relations

$$\begin{aligned}
\mathbf{e}_r &= \sin(\theta)\cos(\varphi)\mathbf{e}_x + \sin(\theta)\sin(\varphi)\mathbf{e}_y + \cos(\theta)\mathbf{e}_z \\
\mathbf{e}_\theta &= \cos(\theta)\cos(\varphi)\mathbf{e}_x + \cos(\theta)\sin(\varphi)\mathbf{e}_y - \sin(\theta)\mathbf{e}_z \\
\mathbf{e}_\varphi &= -\sin(\varphi)\mathbf{e}_x + \cos(\varphi)\mathbf{e}_y,
\end{aligned} \tag{2.95}$$

then we can write the dipole electric field vector as

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 r^3} (3p_r \mathbf{e}_r - \mathbf{p}). \tag{2.96}$$

If the dipole is located at  $\mathbf{x}_0$  (which we now consider to be the origin of the coordinate system), with  $r = |\mathbf{x} - \mathbf{x}_0|$  and  $\mathbf{n}$  the unit vector linking  $\mathbf{x}_0$  to  $\mathbf{x}$  (we could use  $\mathbf{e}_r$  instead of  $\mathbf{n}$ ), we can write the dipole electric field vector as follows

$$\mathbf{E}(\mathbf{x}) = \frac{3\mathbf{n}(\mathbf{n} \cdot \mathbf{p}) - \mathbf{p}}{4\pi\epsilon_0 |\mathbf{x} - \mathbf{x}_0|^3}. \tag{2.97}$$

It is important to note that in general *the multipole moments*  $q_{lm}$  *depend on the choice of the origin.*

Coming back to the evaluation of the electric field due to the dipole moment term, we note that if we calculate the average value of the electric field inside a sphere of radius  $R$  centered at  $\mathbf{x}_0$ , we find that

$$\int_{r < R} \mathbf{E}(\mathbf{x}) d^3x = \int_{r < R} [E_r \mathbf{e}_r + E_\theta \mathbf{e}_\theta + E_\varphi \mathbf{e}_\varphi] r^2 \sin(\theta) dr d\theta d\varphi, \tag{2.98}$$

for which, according to equations (2.93)-(2.94), each term will contain only terms proportional to one of the following integrals

$$\int_0^{2\pi} \sin(\varphi) d\varphi, \quad \text{or} \quad \int_0^{2\pi} \cos(\varphi) d\varphi, \quad \text{or} \quad \int_0^\pi \sin(\theta)\cos(\theta) d\theta. \tag{2.99}$$

Since all of these integrals vanish, we find that the contribution of the dipole term to the volume-averaged electric field also vanishes. That is,

$$\int_{r < R} \mathbf{E}(\mathbf{x}) d^3x = 0. \tag{2.100}$$

This result may not be surprising, since this would be expected if one considers a simple dipole configuration (i.e., two point charges of opposite polarity displaced equally far on either side of  $\mathbf{x}_0$ ) and integrate over the sphere centered on the dipole. Things are different, however, if we consider an arbitrary charge distribution  $\rho(\mathbf{x}')$ . We again

consider a sphere of radius  $R$  centered on the origin of the coordinate system, and which contains the whole charge distribution. We start by writing

$$\begin{aligned}\int_{r < R} \mathbf{E}(\mathbf{x}) d^3x &= - \int_{r < R} \nabla \Phi d^3x \\ &= - \int_{r=R} \Phi(\mathbf{x}) \mathbf{n} R^2 d\Omega,\end{aligned}\tag{2.101}$$

where equation (1.32) was used, and  $\mathbf{n} = \mathbf{x}/R$  is the outward unit vector normal to the surface of the sphere. Replacing the potential  $\Phi$  with its integral form, we have

$$\int_{r < R} \mathbf{E}(\mathbf{x}) d^3x = - \frac{R^2}{4\pi\epsilon_0} \int d^3x' \rho(\mathbf{x}') \int_{r=R} \frac{\mathbf{n}}{|\mathbf{x} - \mathbf{x}'|} d\Omega.\tag{2.102}$$

Since the unit vector can be written as a function of the angles with

$$\mathbf{n} = \sin(\theta)\cos(\varphi)\mathbf{e}_x + \sin(\theta)\sin(\varphi)\mathbf{e}_y + \cos(\theta)\mathbf{e}_z,\tag{2.103}$$

each component  $n_i$  of  $\mathbf{n}$  can be expanded in a series of spherical harmonics where  $l = 1$

$$n_i \propto \sum_{m=-1}^1 c_m Y_{1m}(\theta, \varphi).\tag{2.104}$$

Because of this, each angular integral on the right-hand side of equation (2.102) can be transformed to

$$\begin{aligned}\int_{r=R} \frac{n_i}{|\mathbf{x} - \mathbf{x}'|} d\Omega &\propto \sum_{m=-1}^1 c_m \int_{r=R} \frac{Y_{1m}(\theta, \varphi)}{|\mathbf{x} - \mathbf{x}'|} d\Omega \\ &\propto \sum_{m=-1}^1 c_m \int_{r=R} \sum_{l=0}^{\infty} \frac{r_{<}^l}{r_{>}^{l+1}} P_l[\cos(\gamma)] Y_{1m}(\theta, \varphi) d\Omega \\ &\propto \sum_{m=-1}^1 \sum_{l=0}^{\infty} c_m \frac{r_{<}^l}{r_{>}^{l+1}} \int_{r=R} Y_{l0}(\theta, \varphi) Y_{1m}(\theta, \varphi) d\Omega,\end{aligned}\tag{2.105}$$

where we used equations (2.60) and (2.39), and

$$\cos(\gamma) = \cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\varphi - \varphi').\tag{2.106}$$

Upon using the orthogonality property of spherical harmonics (i.e., equation (2.32)), we find that the only term allowed is that where  $l=1$  in the expansion for  $1/|\mathbf{x} - \mathbf{x}'|$ . Equation (2.102) can therefore be written as

$$\int_{r < R} \mathbf{E}(\mathbf{x}) d^3x = -\frac{R^2}{4\pi\epsilon_0} \int d^3x' \rho(\mathbf{x}') \frac{r_{<}}{r_{>}^2} \int_{r=R} \mathbf{n} \cos(\gamma) d\Omega. \quad (2.107)$$

The angular integral can now be calculated as follows

$$\begin{aligned} \int_{r=R} \mathbf{n} \cos(\gamma) d\Omega &= \int_0^{2\pi} d\varphi \int_0^\pi \left[ \sin(\theta) \cos(\varphi) \mathbf{e}_x + \sin(\theta) \sin(\varphi) \mathbf{e}_y + \cos(\theta) \mathbf{e}_z \right] \\ &\quad \times \left[ \cos(\theta) \cos(\theta') + \sin(\theta) \sin(\theta') \cos(\varphi - \varphi') \right] \sin(\theta) d\theta \\ &= \pi \sin(\theta') \left[ \cos(\varphi') \mathbf{e}_x + \sin(\varphi') \mathbf{e}_y \right] \int_0^\pi \sin^3(\theta) d\theta \\ &\quad + 2\pi \cos(\theta') \mathbf{e}_z \int_0^\pi \sin(\theta) \cos^2(\theta) d\theta \\ &= \frac{4\pi}{3} \left[ \sin(\theta') \cos(\varphi') \mathbf{e}_x + \sin(\theta') \sin(\varphi') \mathbf{e}_y + \cos(\theta') \mathbf{e}_z \right] \\ &= \frac{4\pi}{3} \mathbf{n}', \end{aligned} \quad (2.108)$$

where the last two of equations (2.108) serve as an explicit definition of the unit vector  $\mathbf{n}' = \mathbf{x}'/r'$ . Equation (2.107) is now simplified to

$$\int_{r < R} \mathbf{E}(\mathbf{x}) d^3x = -\frac{R^2}{3\epsilon_0} \int \rho(\mathbf{x}') \frac{r_{<}}{r_{>}^2} \mathbf{n}' d^3x'. \quad (2.109)$$

With the original assumption that the charge distribution is entirely contained within the sphere, we can write  $r_{<} = r'$  and  $r_{>} = R$ , and equation (2.109) reduces to

$$\begin{aligned} \int_{r < R} \mathbf{E}(\mathbf{x}) d^3x &= -\frac{R^2}{3\epsilon_0} \int \rho(\mathbf{x}') \frac{r'}{R^2} \mathbf{n}' d^3x' \\ &= -\frac{1}{3\epsilon_0} \int \mathbf{x}' \rho(\mathbf{x}') d^3x' \\ &= -\frac{\mathbf{p}}{3\epsilon_0}, \end{aligned} \quad (2.110)$$

from equation (2.87). This result states that the average value of the electric field over a volume containing the charge distribution responsible for the field is proportional to the dipole moment of the charge distribution with respect to the center of the sphere. It can be combined to the previous result expressed in equation (2.97) to yield

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \left[ \frac{3\mathbf{n}(\mathbf{n} \cdot \mathbf{p}) - \mathbf{p}}{|\mathbf{x} - \mathbf{x}_0|^3} - \frac{4\pi}{3} \mathbf{p} \delta(\mathbf{x} - \mathbf{x}_0) \right]. \quad (2.111)$$

Equation (2.111) does not change the value of the electric field as calculated with equation (2.97) when  $\mathbf{x} \neq \mathbf{x}_0$ , but it correctly takes into account the required volume integral (2.110).

Interestingly, if we consider a sphere to which the charge distribution is external, we have  $r_> = r'$  and  $r_< = R$  in equation (2.109), and

$$\begin{aligned}
 \int_{r < R} \mathbf{E}(\mathbf{x}) d^3x &= -\frac{R^3}{3\epsilon_0} \int \rho(\mathbf{x}') \frac{\mathbf{n}'}{r'^2} d^3x' \\
 &= -\frac{R^3}{3\epsilon_0} \int \rho(\mathbf{x}') \frac{\mathbf{x}'}{r'^3} d^3x' \\
 &= \frac{R^3}{3} \left[ \int \rho(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} d^3x' \right]_{\mathbf{x}=0} \\
 &= \frac{4\pi R^3}{3} \mathbf{E}(0),
 \end{aligned} \tag{2.112}$$

where we used Coulomb's Law (i.e., equation (1.47)) for the last step. In other words, *the average value of the electric field over a sphere containing no charge is the value of the field at the centre of the sphere.*

## 2.4 Electrostatic Potential Energy

We know from the calculations that led to equation (1.60), on page 11, that the product of the potential and the charge can be interpreted as the potential energy of the charge as it is brought from infinity (where the potential is assumed to vanish) to its final position. More precisely, we define the potential energy  $W_i$  of a charge  $q_i$  with

$$W_i = q_i \Phi(\mathbf{x}_i). \tag{2.113}$$

If the potential is due to an ensemble of  $n - 1$  charges  $q_j$ , then the potential energy of the charge  $q_i$  becomes

$$W_i = \frac{q_i}{4\pi\epsilon_0} \sum_{j=1}^{n-1} \frac{q_j}{|\mathbf{x}_i - \mathbf{x}_j|}, \tag{2.114}$$

and the total potential energy  $W$  is

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \sum_{j < i} \frac{q_i q_j}{|\mathbf{x}_i - \mathbf{x}_j|}. \tag{2.115}$$

Alternatively, we can write a more symmetric equation for the total energy by summing over all charges and dividing by two

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^n \sum_{j \neq i}^n \frac{q_i q_j}{|\mathbf{x}_i - \mathbf{x}_j|}. \quad (2.116)$$

Equation (2.116) can easily be generalized to continuous charge distributions with

$$W = \frac{1}{8\pi\epsilon_0} \iint \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x d^3x' \quad (2.117)$$

or, alternatively, if we use equation (1.57) for the potential

$$W = \frac{1}{2} \int \rho(\mathbf{x})\Phi(\mathbf{x}) d^3x \quad (2.118)$$

Furthermore, we can express the potential energy using the electric field instead of the charge distribution and the potential. To so, we replace the potential with the Poisson equation in equation (2.118) and proceed as follows

$$\begin{aligned} W &= -\frac{\epsilon_0}{2} \int \Phi \nabla^2 \Phi d^3x \\ &= -\frac{\epsilon_0}{2} \int \left[ \nabla \cdot (\Phi \nabla \Phi) - (\nabla \Phi)^2 \right] d^3x \\ &= -\frac{\epsilon_0}{2} \left[ \int_S (\Phi \nabla \Phi) \cdot \mathbf{n} da - \int |\mathbf{E}|^2 d^3x \right], \end{aligned} \quad (2.119)$$

where we used equation (1.26) and the divergence theorem for the second and last lines, respectively. Because the integration is done over all of space, the surface in integral will vanish since

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_S (\Phi \nabla \Phi) \cdot \mathbf{n} da &\propto \lim_{R \rightarrow \infty} \int_S \frac{1}{R} \cdot \frac{1}{R^2} R^2 d\Omega \\ &\propto \lim_{R \rightarrow \infty} \frac{4\pi}{R} = 0. \end{aligned} \quad (2.120)$$

The potential energy then becomes

$$W = \frac{\epsilon_0}{2} \int |\mathbf{E}|^2 d^3x \quad (2.121)$$

It follows from this result that the **potential energy density** is defined as

$$w = \frac{\epsilon_0}{2} |\mathbf{E}|^2. \quad (2.122)$$

It is important to note that the energy calculated with equation (2.121) will be greater than that evaluated using equation (2.117) as it contains (that is, equation (2.121)) the “self-energy” of the charge distribution.

We are now interested in expressing the potential energy of a charge distribution subjected to an external field as consisting of the contributions from the different terms in the multipole expansion. In this case, we know from equation (2.113) that

$$W = \int \rho(\mathbf{x}) \Phi(\mathbf{x}) d^3x. \quad (2.123)$$

We start by expanding the potential function with a Taylor series (see equation (1.84)) around some predefined origin

$$\begin{aligned} \Phi(\mathbf{x}) &= \Phi(0) + \mathbf{x} \cdot \nabla \Phi \Big|_{\mathbf{x}=0} + \frac{1}{2} x_i x_j \frac{\partial^2 \Phi}{\partial x_i \partial x_j} \Big|_{\mathbf{x}=0} + \dots \\ &= \Phi(0) - \mathbf{x} \cdot \mathbf{E}(0) - \frac{1}{2} x_i x_j \frac{\partial E_j}{\partial x_i} \Big|_{\mathbf{x}=0} + \dots, \end{aligned} \quad (2.124)$$

where we used  $\mathbf{E} = -\nabla \Phi$ , and summations over repeated indices were implied. Since  $\nabla \cdot \mathbf{E} = 0$  for the external field, we can subtract the following relation from the last term of equation (2.124) without effectively changing anything

$$\frac{1}{6} r^2 \nabla \cdot \mathbf{E} \Big|_{\mathbf{x}=0} = \frac{1}{6} r^2 \delta_{ij} \frac{\partial E_j}{\partial x_i} \Big|_{\mathbf{x}=0} \quad (2.125)$$

to get

$$\Phi(\mathbf{x}) = \Phi(0) - \mathbf{x} \cdot \mathbf{E}(0) - \frac{1}{6} (3x_i x_j - r^2 \delta_{ij}) \frac{\partial E_j}{\partial x_i} \Big|_{\mathbf{x}=0} + \dots \quad (2.126)$$

If we insert equation (2.126) into equation (2.118) for the potential energy, and using equation (2.88) for the components of the quadrupole moment tensor, we get

$$\boxed{W = q\Phi(0) - \mathbf{p} \cdot \mathbf{E}(0) - \frac{1}{6} Q_{ij} \frac{\partial E_j}{\partial x_i} \Big|_{\mathbf{x}=0} + \dots} \quad (2.127)$$

Using equation (2.111) for the electric field generated by a dipole, and equation (2.127), we can calculate the energy of interaction between two dipoles  $\mathbf{p}_1$  and  $\mathbf{p}_2$  located at  $\mathbf{x}_1$  and  $\mathbf{x}_2$ , respectively, when  $\mathbf{x}_1 \neq \mathbf{x}_2$ . Thus,

$$\begin{aligned} W_{12} &= -\mathbf{p}_1 \cdot \mathbf{E}_2(\mathbf{x}_1) = -\mathbf{p}_2 \cdot \mathbf{E}_1(\mathbf{x}_2) \\ &= \frac{\mathbf{p}_1 \cdot \mathbf{p}_2 - 3(\mathbf{p}_1 \cdot \mathbf{n})(\mathbf{p}_2 \cdot \mathbf{n})}{4\pi\epsilon_0 |\mathbf{x}_1 - \mathbf{x}_2|^3}, \end{aligned} \quad (2.128)$$

where  $\mathbf{n} = (\mathbf{x}_1 - \mathbf{x}_2)/|\mathbf{x}_1 - \mathbf{x}_2|$ . The dipoles are attracted to each other when the energy is negative, and vice-versa. For example, when the dipoles are parallel in their orientation, and to the line joining them, then,  $W < 0$  and they will attract each other.

## 2.5 Electrostatic fields in Matter

So far in dealing with the equations of electrostatic, we were concerned with, and derived, the *microscopic equations of electrostatic*. That is, we considered problems involving mainly charge distributions without the presence of any ponderable media. When analyzing electrostatic fields in matter, we need to make averages over macroscopically small, but microscopically large, volumes to obtain the *macroscopic equations of electrostatics*.

In the first place, if we think of the macroscopic electric field  $\mathbf{E}$  at a given point  $\mathbf{x}$  as some average of the microscopic electric field  $\mathbf{E}_\mu$  over some surrounding volume  $\Delta V$ , we can write

$$\mathbf{E}(\mathbf{x}) = \frac{1}{\Delta V} \int_{\Delta V} \mathbf{E}_\mu(\mathbf{x} + \mathbf{x}') d^3x'. \quad (2.129)$$

If we calculate the curl of the macroscopic field, we have

$$\begin{aligned} \nabla \times \mathbf{E}(\mathbf{x}) &= \nabla \times \left[ \frac{1}{\Delta V} \int_{\Delta V} \mathbf{E}_\mu(\mathbf{x} + \mathbf{x}') d^3x' \right] \\ &= \frac{1}{\Delta V} \int_{\Delta V} \nabla \times \mathbf{E}_\mu(\mathbf{x} + \mathbf{x}') d^3x', \end{aligned} \quad (2.130)$$

and since from the microscopic equation  $\nabla \times \mathbf{E}_\mu = 0$ , then

$$\nabla \times \mathbf{E} = 0 \quad (2.131)$$

for the macroscopic field. So the same relation between the electric field and the potential exists when dealing with ponderable media. That is,

$$\mathbf{E}(\mathbf{x}) = -\nabla\Phi. \quad (2.132)$$

When a medium made up atoms or molecules is subjected to an external electric field, the charges making up the molecules will react to its presence by individually producing a, or enhancing an already existing, dipole moment. The material as a whole will, therefore, become electrically polarized, the resulting dipole moment being the dominant multipole term. That is, an **electric polarization**  $\mathbf{P}$  (dipole moment per unit volume) given by

$$\mathbf{P}(\mathbf{x}) = \sum_i N_i(\mathbf{x}) \mathbf{p}_i \quad (2.133)$$

is induced in the medium, where  $\mathbf{p}_i$  is the average dipole moment of the  $i$ th type of molecules in the medium (calculated in the same manner as the electric field was in equation (2.129)), and  $N_i$  is the average number density of the same type of molecules at point  $\mathbf{x}$ . The average charge density  $\rho(\mathbf{x})$  is evaluated using the same process.

Similarly, we can use equation (2.89) to calculate the contribution to the potential at the position  $\mathbf{x}$  from the macroscopically small volume element  $\Delta V$  located at  $\mathbf{x}'$

$$\Delta\Phi(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi\epsilon_0} \left[ \frac{\rho(\mathbf{x}')\Delta V}{|\mathbf{x} - \mathbf{x}'|} + \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{P}(\mathbf{x}')\Delta V}{|\mathbf{x} - \mathbf{x}'|^3} \right], \quad (2.134)$$

where  $\rho(\mathbf{x}')\Delta V$  and  $\mathbf{P}(\mathbf{x}')\Delta V$  are, respectively, the average monopole (or charge) and dipole moments contained in the volume. Taking the limit  $\Delta V \rightarrow d^3x'$ , and integrate over all space, we get the potential

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \left[ \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \mathbf{P}(\mathbf{x}') \cdot \nabla' \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right], \quad (2.135)$$

where we used equation (1.55). Transforming the second term, we find

$$\begin{aligned} \int \mathbf{P}(\mathbf{x}') \cdot \nabla' \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) d^3x' &= \int \left[ \nabla' \cdot \left( \frac{\mathbf{P}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right) - \frac{\nabla' \cdot \mathbf{P}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right] d^3x' \\ &= \int \frac{\mathbf{P}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \cdot \mathbf{n}' da' - \int \frac{\nabla' \cdot \mathbf{P}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \\ &= - \int \frac{\nabla' \cdot \mathbf{P}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x', \end{aligned} \quad (2.136)$$

since the first integral on the right-hand side is over an infinite surface. The potential now becomes

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} [\rho(\mathbf{x}') - \nabla' \cdot \mathbf{P}(\mathbf{x}')]. \quad (2.137)$$

Equation (2.137) has the same form as equation (1.57) for the potential, as long as we define a new “effective” charge density  $(\rho - \nabla \cdot \mathbf{P})$ . By effective charge density, we mean that if the polarization is non-uniform in a given region (i.e.,  $\nabla \cdot \mathbf{P} \neq 0$ ), then there will be a net change in the amount of charge within that region. Upon using equation (2.132), we can write

$$\begin{aligned}
 \nabla \cdot \mathbf{E}(\mathbf{x}) &= -\nabla^2 \Phi \\
 &= -\frac{1}{4\pi\epsilon_0} \int \nabla^2 \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) [\rho(\mathbf{x}') - \nabla' \cdot \mathbf{P}(\mathbf{x}')] d^3x' \\
 &= \frac{1}{\epsilon_0} \int \delta(\mathbf{x} - \mathbf{x}') [\rho(\mathbf{x}') - \nabla' \cdot \mathbf{P}(\mathbf{x}')] d^3x' \\
 &= \frac{1}{\epsilon_0} [\rho(\mathbf{x}) - \nabla \cdot \mathbf{P}(\mathbf{x})].
 \end{aligned} \tag{2.138}$$

We now define the **electric displacement  $\mathbf{D}$**  as

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}} \tag{2.139}$$

and equation (2.138) becomes

$$\boxed{\nabla \cdot \mathbf{D} = \rho} \tag{2.140}$$

In most media  $\mathbf{P}$  is proportional to  $\mathbf{E}$  (i.e., the media are *linear* and *isotropic*), and we write

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \tag{2.141}$$

where  $\chi_e$  is the **electric susceptibility** of the medium. In such cases,  $\mathbf{D}$  is also proportional to  $\mathbf{E}$  and

$$\mathbf{D} = \epsilon \mathbf{E}, \tag{2.142}$$

with the **electric permittivity  $\epsilon$**  defined by

$$\epsilon = \epsilon_0 (1 + \chi_e). \tag{2.143}$$

The quantity  $\epsilon/\epsilon_0$  is called the **dielectric constant**. If the medium is also *homogeneous*, then the divergence of the electric field becomes

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon}. \quad (2.144)$$

Consequently, any electrostatic problem in a *linear, isotropic, and homogeneous* medium is equivalent to one set in vacuum (solved using the microscopic equations) as long as the electric field is scale by a factor  $\epsilon_0/\epsilon$ . Finally, the boundary conditions derived for the microscopic electric field (see equations (1.79)) can easily be extended to such medium

$$\begin{aligned} (\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n} &= \sigma \\ (\mathbf{E}_2 - \mathbf{E}_1) \times \mathbf{n} &= 0, \end{aligned} \quad (2.145)$$

where  $\mathbf{n}$  is a unit vector extending from medium 1 to medium 2, and normal to the boundary surface, and  $\sigma$  is the surface charge.

## 2.6 Electrostatic Energy in Dielectric Media

In evaluating the amount of energy contained in a dielectric, we must be cautious to carefully include not only the energy needed to assemble the charge distribution (by bringing it from infinity), as was done in section 2.4, but also the energy spent in polarizing the medium. So, if we consider a change  $\delta\rho$  in the macroscopic charge distribution, which extends over all space, then the work done to accommodate this change is (see equation (2.113))

$$\delta W = \int \delta\rho(\mathbf{x})\Phi(\mathbf{x})d^3x, \quad (2.146)$$

where  $\Phi$  is the potential due to the already existing charge density  $\rho(\mathbf{x})$ . Using equation (2.140), we also write the change in charge density as

$$\delta\rho = \nabla \cdot (\delta\mathbf{D}). \quad (2.147)$$

Inserting this result in equation (2.146) we get

$$\begin{aligned} \delta W &= \int [\nabla \cdot (\delta\mathbf{D})] \Phi d^3x \\ &= \int [\nabla \cdot (\delta\mathbf{D}\Phi) - (\delta\mathbf{D}) \cdot \nabla\Phi] d^3x \\ &= \int \mathbf{E} \cdot \delta\mathbf{D} d^3x, \end{aligned} \quad (2.148)$$

since the integral of the divergence on the second line vanishes when transformed into a surface integral (over all space) using the divergence theorem. Initially, before the dielectric medium is assembled and the electric density attains its final value  $\mathbf{D}$ , we must have  $\mathbf{D} = 0$ . Therefore,

$$W = \int d^3x \int_0^{\mathbf{D}} \mathbf{E} \cdot \delta \mathbf{D}. \quad (2.149)$$

If the medium is linear, then  $\mathbf{D} = \varepsilon \mathbf{E}$  and

$$\delta(\mathbf{E} \cdot \mathbf{D}) = \varepsilon \delta(\mathbf{E} \cdot \mathbf{E}) = 2\varepsilon(\mathbf{E} \cdot \delta \mathbf{E}) = 2(\mathbf{E} \cdot \delta \mathbf{D}), \quad (2.150)$$

and the total electrostatic energy in the medium is

$$\boxed{W = \frac{1}{2} \int \mathbf{E} \cdot \mathbf{D} d^3x} \quad (2.151)$$

Finally, let's consider a dielectric  $\varepsilon_1$  immersed in another (with permittivity  $\varepsilon_0$ ) where a fixed electric field  $\mathbf{E}_0$  exists. We assume that there are no free charge distributions. That is

$$\nabla \cdot \mathbf{D} = \nabla \cdot \mathbf{D}_0 = 0. \quad (2.152)$$

If the first dielectric  $\varepsilon_1$  were absent, the energy contained within the equivalent volume of space  $V_1$  it occupies would be

$$W_0 = \frac{1}{2} \int \mathbf{E}_0 \cdot \mathbf{D}_0 d^3x, \quad (2.153)$$

whereas it is

$$W_1 = \frac{1}{2} \int \mathbf{E} \cdot \mathbf{D} d^3x \quad (2.154)$$

with it in place. The difference between the two energies is

$$\begin{aligned} W \equiv W_1 - W_0 &= \frac{1}{2} \int_{V_1} (\mathbf{E} \cdot \mathbf{D} - \mathbf{E}_0 \cdot \mathbf{D}_0) d^3x \\ &= \frac{1}{2} \int_{V_1} (\mathbf{E} \cdot \mathbf{D}_0 - \mathbf{E}_0 \cdot \mathbf{D}) d^3x + \frac{1}{2} \int_{V_1} (\mathbf{E} + \mathbf{E}_0) \cdot (\mathbf{D} - \mathbf{D}_0) d^3x. \end{aligned} \quad (2.155)$$

However, since  $\nabla \times (\mathbf{E} + \mathbf{E}_0) = 0$  and, therefore,  $\mathbf{E} + \mathbf{E}_0 = -\nabla \Phi$ , and  $\nabla \cdot (\mathbf{D} - \mathbf{D}_0) = 0$  from equation (2.152), the second integral on the right-hand side of equation (2.155) is shown to vanish from

$$\begin{aligned}
\frac{1}{2} \int_{V_1} (\mathbf{E} + \mathbf{E}_0) \cdot (\mathbf{D} - \mathbf{D}_0) d^3x &= -\frac{1}{2} \int_{V_1} \nabla \Phi \cdot (\mathbf{D} - \mathbf{D}_0) d^3x \\
&= -\frac{1}{2} \int_{V_1} \left\{ \nabla \cdot [(\mathbf{D} - \mathbf{D}_0) \Phi] - \Phi \nabla \cdot (\mathbf{D} - \mathbf{D}_0) \right\} d^3x \\
&= \frac{1}{2} \int_{V_1} \nabla \cdot [(\mathbf{D} - \mathbf{D}_0) \Phi] d^3x \\
&= \frac{1}{2} \int_{S_1} \Phi (\mathbf{D} - \mathbf{D}_0) \cdot \mathbf{n} da \\
&= 0,
\end{aligned} \tag{2.156}$$

where the integral of the divergence term was transformed into a surface integral over  $S_1$ , the surface delimiting the volume  $V_1$ , which vanishes because  $\mathbf{D} = \mathbf{D}_0$  at the surface (or just beyond it). Hence,

$$W = \frac{1}{2} \int_{V_1} (\mathbf{E} \cdot \mathbf{D}_0 - \mathbf{E}_0 \cdot \mathbf{D}) d^3x. \tag{2.157}$$

Alternatively, if we use  $\mathbf{D} = \varepsilon_1 \mathbf{E}$  and  $\mathbf{D}_0 = \varepsilon_0 \mathbf{E}_0$ , and the region occupied by the object belongs to free space (i.e., the permittivity  $\varepsilon_0$  is that of vacuum), then

$$\begin{aligned}
W &= -\frac{1}{2} \int_{V_1} (\varepsilon_1 - \varepsilon_0) \mathbf{E} \cdot \mathbf{E}_0 d^3x. \\
&= -\frac{1}{2} \int_{V_1} \mathbf{P} \cdot \mathbf{E}_0 d^3x.
\end{aligned} \tag{2.158}$$

We can, therefore, define the energy density of a dielectric in a fixed electric field  $\mathbf{E}_0$  as

$$w = -\frac{1}{2} \mathbf{P} \cdot \mathbf{E}_0. \tag{2.159}$$

If we allow for an *increase*  $\delta E_0$  in the electric field relative to some coordinate  $x'_i$ , then equation (2.158) tells us that there will be a corresponding *decrease*  $\delta W$  in the potential energy. We, therefore, find that the force on the dielectric is

$$F = - \lim_{\delta E_0 \rightarrow 0} \frac{\delta W}{\delta x'_i} > 0, \tag{2.160}$$

and that the dielectric will tend to accelerate toward regions of increasing electric field intensity provided that  $\varepsilon_1 > \varepsilon_0$ .